THE REAL FIELD WITH AN IRRATIONAL POWER FUNCTION AND A DENSE MULTIPLICATIVE SUBGROUP

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Abstract. This paper provides a first example of a model theoretically well behaved structure consisting of a proper o-minimal expansion of the real field and a dense multiplicative subgroup of finite rank. Under certain Schanuel conditions, a quantifier elimination result will be shown for the real field with an irrational power function x^τ and a dense multiplicative subgroup of finite rank whose elements are algebraic over $\mathbb{Q}(\tau)$. Moreover, every open set definable in this structure is already definable in the reduct given by just the real field and the irrational power function.

1. Introduction

Let $\tau \in \mathbb{R} \setminus \mathbb{Q}$. We will consider the multiplicative group $(\mathbb{R}_{>0}, \cdot)$ as a $\mathbb{Q}(\tau)$-linear space where the multiplication is given by a^q for every $q \in \mathbb{Q}(\tau)$ and $a \in \mathbb{R}_{>0}$.

Schanuel condition. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}^n$, then*

$$td_{\mathbb{Q}(\tau)}(a) + m.dim_{\mathbb{Q}(\tau)}(a) \geq m.dim_{\mathbb{Q}}(a),$$

where $td_{\mathbb{Q}(\tau)}(a)$ is the transcendence degree of a over $\mathbb{Q}(\tau)$, $m.dim_{\mathbb{Q}(\tau)}(a)$ and $m.dim_{\mathbb{Q}}(a)$ are the dimensions of the $\mathbb{Q}(\tau)$- and $\mathbb{Q}$-linear subspaces of $\mathbb{R}_{>0}$ generated by a.

Let $\overline{\mathbb{R}} = (\mathbb{R}, <, +, \cdot, 0, 1)$ be the field of real numbers and let x^τ be the function on $\mathbb{R}$ sending t to t^τ for $t > 0$ and to 0 for $t \leq 0$. Let $\mathbb{Q}(\tau)^{ac}$ be the algebraic closure of $\mathbb{Q}(\tau)$. The main result of this paper is the following:

Theorem A. *Let $\tau \in \mathbb{R}$ satisfy the Schanuel condition and let Γ be a dense subgroup of $\mathbb{R}_{>0}$ of finite rank with $\Gamma \subseteq \mathbb{Q}(\tau)^{ac}$. Then every definable set in $(\overline{\mathbb{R}}, x^\tau, \Gamma)$ is a boolean combination of sets of the form*

$$\bigcup_{g \in \Gamma^n} \{x \in \mathbb{R}^m \ : \ (x, g) \in S\},$$

where $S \subseteq \mathbb{R}^{m+n}$ is definable in $(\overline{\mathbb{R}}, x^\tau)$. Moreover, every open set definable in $(\overline{\mathbb{R}}, x^\tau, \Gamma)$ is already definable in $(\overline{\mathbb{R}}, x^\tau)$.

A finite rank subgroup of $\mathbb{R}_{>0}$ is a subgroup that is contained in the divisible closure of a finitely generated subgroup. In fact, we will prove Theorem A not only for finite rank subgroups, but also for subgroups whose divisible closure has the Mann property (see page 4 for a definition of the Mann property). By work of Bays, Kirby and Wilkie in [1] the Schanuel condition holds for co-countably many real numbers τ. Assuming Schanuel's conjecture, the Schanuel condition also holds

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when τ is algebraic (see page 4 for a statement of Schanuel's conjecture).

The significance of Theorem A comes from the fact that it produces the first example of a model theoretically well behaved structure consisting of a proper o-minimal expansion of the real field and a dense multiplicative subgroup of finite rank. So far it was only known by work of van den Dries and Günaydın in [5] that Theorem A holds if $(\overline{\mathbb{R}}, x^\tau)$ is replaced by $\overline{\mathbb{R}}$. In particular, by [7], every open set definable in an expansion of the real field by a dense multiplicative subgroup Γ of $\mathbb{R}_{>0}$ of finite rank is semialgebraic. However Tychonievich showed in [13] that the structure $(\overline{\mathbb{R}}, \Gamma)$ expanded by the restriction of the exponential function to the unit interval defines the set of integers and hence every projective subset of the real line. Such a structure is as wild from a model theoretic view point as it can be. In contrast to this, every expansion of the real field whose open definable sets are definable in an o-minimal expansion, can be considered to be well behaved. For details, see Miller [11] and Dolich, Miller and Steinhorn [3].

None of the assumptions of Theorem A can be dropped. By [8], Corollary 1.5, $(\overline{\mathbb{R}}, x^\tau, 2^{\mathbb{Z}})$ defines the set of integers. For $\tau = \log_2(3)$, the Schanuel condition fails. Since $(\overline{\mathbb{R}}, x^{\log_2(3)}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$ defines $2^{\mathbb{Z}}$, it also defines $\mathbb{Z}$. On the other hand, for a non-algebraic real number τ satisfying the Schanuel condition such that 2^τ is not in $\mathbb{Q}(\tau)^{ac}$, we have again that $2^{\mathbb{Z}}$ is definable $(\overline{\mathbb{R}}, x^\tau, 2^{\mathbb{Z}}2^{\tau\mathbb{Z}})$ and so is $\mathbb{Z}$.

However, Theorem A holds for certain multiplicative subgroups containing elements that are not algebraic over $\mathbb{Q}(\tau)$.

Theorem B. *Let $\tau \in \mathbb{R}$ satisfy the Schanuel condition, let $a_1, ..., a_n \in \mathbb{Q}(\tau)^{ac}$ and let Δ be the $\mathbb{Q}(\tau)$-linear subspace of $(\mathbb{R}_{>0}, \cdot)$ generated by $a_1, ..., a_n$. Then every definable set in $(\overline{\mathbb{R}}, x^\tau, \Delta)$ is a boolean combination of sets of the form*

$$\bigcup_{g \in \Delta^n} \{x \in \mathbb{R}^m \ : \ (x, g) \in S\},$$

where $S \subseteq \mathbb{R}^{m+n}$ is definable in $(\overline{\mathbb{R}}, x^\tau)$. Moreover, every open set definable in $(\overline{\mathbb{R}}, x^\tau, \Delta)$ is already definable in $(\overline{\mathbb{R}}, x^\tau)$.

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1.2. **Coventions and notations.** Above and in the rest of the paper l, m, n always denote natural numbers. Also as usual 'definable' means 'definable with parameters' and when we want to make the language and the parameters explicit we write $\mathfrak{L}$-B-*definable* to mean definable in the appropriate $\mathfrak{L}$-structure using parameters from the set B.

In all instances, K will be either $\mathbb{Q}$ or $\mathbb{Q}(\tau)$ and Γ will always denote a multiplicative subgroup of $\mathbb{R}_{>0}$. Further, every linear space considered in this paper will be a linear subspace of $(M_{>0}, \cdot)$ and *not* of $(M, +)$, where M is a real closed field. In the case that $M_{>0}$ is a K-linear space, we will write $\text{m.dim}_K(S_1/S_0)$ for the K-linear dimension of the quotient linear space of the K-linear space generated by $S_0 \cup S_1$

and S_0, where $S_0, S_1 \subseteq M_{>0}$.

For a given variety W, we will write $\dim W$ for its dimension. For sets X_0, X_1 in a field extending $\mathbb{Q}(\tau)$ we will write $\mathrm{td}_{\mathbb{Q}(\tau)}(X_1/X_0)$ for the transcendence degree of the field extension $\mathbb{Q}(\tau)(X_1 \cup X_0)/\mathbb{Q}(\tau)(X_0)$.

As usual, for any subset $S \subseteq X \times Y$ and $x \in X$, we write $S(x)$ for the set

$$\{y \in Y \ : \ (x, y) \in S\}.$$

For a subset $S \subseteq X^n$, $x \in S$ and a projection $\pi : X^n \to X^l$, we write $S(\pi(x))$ for the set

$$\{y \in S : \pi(y) = \pi(x)\}.$$

1.3. **O-minimality.** Let $\tau \in \mathbb{R} \setminus \mathbb{Q}$ and let x^τ be the function on $\mathbb{R}$ sending t to t^τ for $t > 0$ and to 0 for $t \leq 0$. In this paper we consider the structure $(\overline{\mathbb{R}}, x^\tau, \tau)$. We write T for its theory and $\mathfrak{L}$ for its language. In [10] Miller showed that the theory T is o-minimal and model complete.

In the rest of this paper only the following facts about the o-minimality of T will be used:

Let M be a model of T. A definable subset C of M^n is a *cell*, if for some projection $\pi : M^n \to M^m$, π is a homeomorphism of C onto its image and $\pi(C)$ is open. Since T is o-minimal, every definable set $X \subseteq M^n$ is a finite union of cells which are defined over the same parameter set.

Let A be any subset of M. We write $\mathbf{cl}_T(A)$ for the definable closure of A in M. By o-minimality of T, $\mathbf{cl}_T(A)$ is itself a model of T. Moreover, the function $\mathbf{cl}_T(-)$ is a *pregeometry*; that is for every $A \subseteq M$, $a \in A$ and $b \in M$

(i) $A \subseteq \mathbf{cl}_T(A)$,
(ii) $b \in \mathbf{cl}_T(A)$ iff $b \in \mathbf{cl}_T(A_0)$, for some finite $A_0 \subseteq A$,
(iii) $\mathbf{cl}_T(\mathbf{cl}_T(A)) = \mathbf{cl}_T(A)$,
(iv) if $b \in \mathbf{cl}_T(A) \setminus \mathbf{cl}_T(A \setminus \{a\})$, then $a \in \mathbf{cl}_T\big((A \setminus \{a\}) \cup \{b\}\big)$.

Property (iv) is called *the Steinitz exchange principle.*
For two subsets $A, B \subseteq M$, we will say that A is $\mathbf{cl}_T$*-independent* over B if for every $a \in A$

$$a \notin \mathbf{cl}_T\big(B \cup (A \setminus \{a\})\big).$$

Let M, M' be two models of T, let $N \preceq M, N' \preceq M'$ and let $\beta : N \to N'$ be an $\mathfrak{L}$-isomorphism. Let $a \in M$ and $b \in M'$ be such that $a < c$ iff $b < \beta(c)$, for all $c \in N$. Then there is an $\mathfrak{L}$-isomorphism $\beta' : \mathbf{cl}_T(N, a) \to \mathbf{cl}_T(N', b)$ extending β and sending a to b.

2. A Schanuel condition and the Mann property

Let $\tau \in \mathbb{R}$. As above, we will consider $(\mathbb{R}_{>0}, \cdot)$ as a $\mathbb{Q}(\tau)$-linear space. For $a \in \mathbb{R}^n_{>0}$, we write $\mathrm{m.dim}_{\mathbb{Q}(\tau)}(a)$ and $\mathrm{m.dim}_{\mathbb{Q}}(a)$ for the dimensions of the $\mathbb{Q}(\tau)$- and $\mathbb{Q}$-linear subspaces of $\mathbb{R}_{>0}$ generated by a.

Condition 2.1. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}^n$, then*

$$td_{\mathbb{Q}(\tau)}(a) + m.dim_{\mathbb{Q}(\tau)}(a) \geq m.dim_{\mathbb{Q}}(a).$$

This condition has been analysed in [1]. The main theorem of [1] states that Condition 2.1 holds for co-countably many real numbers.

Theorem 2.2. *([1] Theorem 1.3) Let $\tau \in \mathbb{R}$. If τ is not $\emptyset$-definable in $(\overline{\mathbb{R}}, \exp)$, then Condition 2.1 holds.*

It is not known whether there is any other irrational number τ such that Condition 2.1 holds. However it follows easily from a famous open conjecture of Schanuel, that every *algebraic* real number τ satisfies Condition 2.1. The conjecture states as follows.

Conjecture 2.3. *(Schanuel's Conjecture) Let $n \in \mathbb{N}$ and $a \in \mathbb{R}^n$, then*

$$td_{\mathbb{Q}}(a, \exp(a)) \geq m.dim_{\mathbb{Q}}(\exp(a)).$$

2.1. **The Mann property.** In this section we consider the Mann property and its connection to Condition 2.1. Let F be a field, E be a subfield of F and G be any subgroup of the multiplicative group $F^{\times}$. Consider equations of the form

$$a_1x_1 + ... + a_nx_n = 1, \tag{1}$$

where $a_1, ..., a_n \in E$. We say a solution $(g_1, ..., g_n) \in G^n$ of (1) is *non-degenerate* if for every non-empty subset I of $\{1, ..., n\}$, $\sum_{i \in I} a_ig_i \neq 0$. Further we say that G has the *Mann property over E* if every equation of the above type (1) has only finitely many non-degenerate solutions in G^n. We also call an element $g \in G^n$ a *Mann solution of G over E* if it is a non-degenerate solution in G^n of an equation of the form (1).

In fact, it follows from work of Evertse in [6] and van der Poorten and Schlickewei in [12] that every finite rank multiplicative subgroup of a field of characteristic 0 has the Mann property over $\mathbb{Q}$. Combining this with [5], Proposition 5.6, we get the following theorem.

Theorem 2.4. *Every finite rank multiplicative subgroup of $\mathbb{R}_{>0}$ has the Mann property over $\mathbb{Q}(\tau)$.*

We conclude this section by showing that under Condition 2.1 the $\mathbb{Q}(\tau)$-linear space generated by a *divisible* multiplicative subgroup Γ has the Mann property over $\mathbb{Q}(\tau)$, if Γ has the Mann property over $\mathbb{Q}(\tau)$ and every element of Γ is algebraic over $\mathbb{Q}(\tau)$.

Proposition 2.5. *Assume Condition 2.1 holds for τ. Let Γ be a divisible multiplicative subgroup of $\mathbb{R}_{>0}$ with $\Gamma \subseteq \mathbb{Q}(\tau)^{ac}$ and Δ be the $\mathbb{Q}(\tau)$-linear subspace of $\mathbb{R}_{>0}$ generated by Γ. Then*

(i) *every Mann solution of Δ over $\mathbb{Q}(\tau)$ is in Γ and*
(ii) *Δ has the Mann property over $\mathbb{Q}(\tau)$, if Γ has the Mann property over $\mathbb{Q}(\tau)$.*

Proof. It is enough to show (i). Therefor let $a_1, ..., a_n \in \mathbb{Q}(\tau)$ and let $g = (g_1, ..., g_n) \in \Delta^n$ be such that

$$a_1g_1 + ... + a_ng_n = 1 \tag{2}$$

and for all $I \subseteq \{1, ..., n\}$

$$\sum_{i \in I} a_ig_i \neq 0. \tag{3}$$

We will show that $g \in \Gamma^n$.
Let $h \in \Gamma^m$ be such that $\mathrm{m.dim}_{\mathbb{Q}(\tau)}(g/h) = 0$ and $\mathrm{m.dim}_{\mathbb{Q}}(h) = m$. Let k be the maximal natural number such that there is a subtuple g' of g of length k such that $\mathrm{m.dim}_{\mathbb{Q}}(g'/h) = k$. It just remains to verify that $k = 0$. For a contradiction, suppose that $k > 0$. By (2) and (3), we have that

$$\mathrm{td}_{\mathbb{Q}(\tau)}(g'/h) < k.$$

Since every coordinate of h is algebraic over $\mathbb{Q}(\tau)$,

$$\mathrm{td}_{\mathbb{Q}(\tau)}(h, g') + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(h, g') < k + m = \mathrm{m.dim}_{\mathbb{Q}}(h, g').$$

This contradicts Condition 2.1. □

3. Tori and special pairs

Let M be a model of T. In the following we will consider $(M_{>0}, \cdot)$ as a K-linear space where K is either $\mathbb{Q}$ or $\mathbb{Q}(\tau)$ and the multiplication is given by a^q for every $q \in K$ and $a \in M_{>0}$.

Definition 3.1. *A basic K-*torus *L_0 of M^n is the set of solutions of equations of the form*

$$\begin{aligned} x_1^{p_{1,1}} \cdot \ldots \cdot x_n^{p_{1,n}} &= 1, \\ \vdots \qquad & \vdots \\ x_1^{p_{l,1}} \cdot \ldots \cdot x_n^{p_{l,n}} &= 1, \end{aligned}$$

where $p_{i,j} \in K$.
*For $b \in M^m$, a K-*torus *L over b is a subset of $M^n_{>0}$ of the form $L_0(b)$, for some basis K-torus L_0 of $M^{m+n}_{>0}$. We will write $\dim L$ for the dimension of L which is given by the corank of the matrix $(p_{i,j})_{i=1,\ldots,l,j=m+1,\ldots,m+n}$.*

The dimension of a torus and the linear dimension of a tuple in $M_{>0}$ correspond to each other. Let $a \in M^n$, $b \in M^m$ and let L be the minimal $\mathbb{Q}(\tau)$-torus over b containing a. Then

$$\dim L = \mathrm{m.dim}_{\mathbb{Q}(\tau)}(a/b).$$

For the following, the reader is reminded that for a set $S \subseteq M^n$, $y \in S$ and a projection $\pi : M^n \to M^l$ we write $S(\pi(y))$ for the set

$$\{z \in S \ : \ \pi(z) = \pi(y)\}.$$

Definition 3.2. *Let $W \subseteq M^n$ be a variety and let L be a $\mathbb{Q}(\tau)$-torus. The pair (W, L) is called* special*, if $n = 0$ or*

$$\dim W(\pi(y)) + \dim L(\pi(y)) < n - l,$$

for every point $y \in W \cap L$ and every projection $\pi : M^n \to M^l$, where $l \in \{0, \ldots, n\}$.

Note that the notion of specialness is first order expressible. In particular, for given variety $W \subseteq M^{m+n}$ and $\mathbb{Q}(\tau)$-torus $L \subseteq M^{m+n}$, the set

$$\{a \in M^m \ : \ (W(a), L(a)) \text{ is special } \}$$

is $\mathfrak{L}$-definable, where $\mathfrak{L}$ is the language of $(\overline{\mathbb{R}}, x^\tau, \tau)$.

3.1. **A Mordell-Lang Theorem for special pairs.** Let Γ be a multiplicative subgroup of $\mathbb{R}_{>0}$ such that the divisible closure of Γ has the Mann property over $\mathbb{Q}(\tau)$ and Γ is a subset of $\mathbb{Q}(\tau)^{ac}$, the algebraic closure of $\mathbb{Q}(\tau)$. Further,

let Δ be the $\mathbb{Q}(\tau)$-linear subspace of $\mathbb{R}_{>0}$ generated by Γ.

In this subsection we will prove the following theorem about special pairs defined over parameters from Δ. Its statement is similar to a conjecture of Mordell and Lang.

Theorem 3.3. *Assume Condition 2.1. Let $W \subsetneq \mathbb{R}^{l+n}$ be a variety defined over $\mathbb{Q}(\tau)$ and let $L \subseteq \mathbb{R}^{l+n}$ be a basic $\mathbb{Q}(\tau)$-torus. Then there are finitely many basic $\mathbb{Q}$-tori $L_1, ..., L_m$ and $g_1, ..., g_m \in \Gamma^{l+n}$ such that*

$$\{(h, y) \in \Delta^l \times \mathbb{R}^n : \ (W(h), L(h)) \text{ is special } \} \cap W \subseteq \bigcup_{i=1}^{m} g_i \cdot L_i. \tag{4}$$

and $\dim L_i(z) < n$, if $n > 0$, for every $z \in \mathbb{R}^l$ and $i = 1, ..., m$.

For the proof of Theorem 3.3 the following lemma is needed.

Lemma 3.4. *Assume Condition 2.1. Let $g \in \Delta^l$, $y \in \mathbb{R}^n$ and let W be a variety defined over $\mathbb{Q}(\tau)(g)$ and L be an $\mathbb{Q}(\tau)$-torus over g. If the pair (W, L) is special and $y \in W \cap L$, then $y \in \Delta^n$.*

Proof. Let $y = (y_1, ..., y_n) \in W \cap L$. If (W, L) is special, we have that for every subset $I \subseteq \{1, ..., n\}$

$$\mathrm{td}_{\mathbb{Q}(\tau)}((y_j)_{j \notin I}/g, (y_i)_{i \in I}) + \mathrm{m.dim}_{\mathbb{Q}(\tau)}((y_j)_{j \notin I}/g, (y_i)_{i \in I}) < n - |I|. \tag{5}$$

For a contradiction suppose that $y \notin \Delta^n$. We easily can reduce to the case that g, y are multiplicatively independent, ie. $\mathrm{m.dim}_{\mathbb{Q}}(g, y) = l + n$. By (5)

$$\mathrm{td}_{\mathbb{Q}(\tau)}(y/g) + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(y/g) < n.$$

By definition of Δ, we can assume there is $s \in \mathbb{N}$ and a subtuple $h \in (\Delta \cap \mathbb{Q}(\tau)^{ac})^s$ of g such that

$$\mathrm{m.dim}_{\mathbb{Q}(\tau)}(g/h) = 0.$$

Hence

$$\begin{aligned} &\mathrm{td}_{\mathbb{Q}(\tau)}(g, y) + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(g, y) \\ &\qquad = \mathrm{td}_{\mathbb{Q}(\tau)}(g) + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(g) + \mathrm{td}_{\mathbb{Q}(\tau)}(y/g) + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(y/g) \\ &\qquad < l - s + s + n = l + n. \end{aligned}$$

By Condition 2.1, $\mathrm{m.dim}_{\mathbb{Q}}(g, y) < l + n$. This is a contradiction to our assumption on g and y. □

In [5] it is shown that the Mann property implies the Mordell-Lang property. In our notation [5], Proposition 5.8, is stated as follows:

Lemma 3.5. *Let G be a multiplicative subgroup of $\mathbb{R}_{>0}$ with the Mann property over $\mathbb{Q}(\tau)$. Then for every variety $W \subseteq \mathbb{R}^n$, there are finitely many basic $\mathbb{Q}$-tori $L_1, ..., L_m$ of $\mathbb{R}^n_{>0}$ and $g_1, ..., g_m \in G^n$ such that*

$$W \cap G^n = \bigcup_{i=1}^{m} g_i \cdot L_i \cap G^n.$$

Moreover, every coordinate of $g_1, ..., g_n$ is a coordinate of a Mann solution of G over $\mathbb{Q}(\tau)$.

The fact that every coordinate of $g_1, ..., g_n$ is a coordinate of a Mann solution over $\mathbb{Q}(\tau)$ is not in the statement of [5], Proposition 5.8, but explicit in its proof.

Proof of Theorem 3.3. Since the divisible closure of Γ has the Mann property over $\mathbb{Q}(\tau)$, Δ has the Mann property over $\mathbb{Q}(\tau)$ by Proposition 2.5(ii). Hence by Lemma 3.5 there are basic $\mathbb{Q}$-tori $L_1, ..., L_m$ and $g_1, ..., g_m \in \Delta^{l+n}$ such that

$$W \cap \Delta^{l+n} = \bigcup_{i=1}^{m} g_i \cdot L_i \cap \Delta^{l+n}. \tag{6}$$

By Proposition 2.5(i), every Mann solution over $\mathbb{Q}(\tau)$ of Δ is in the divisible closure of Γ. Hence every coordinate of $g_1, ..., g_m$ is in the divisible closure of Γ by Lemma 3.5. After changing the L_i's slightly, we can even take $g_1, ..., g_m \in \Gamma^{l+n}$. Finally, the left hand side of (4) in the statement of the theorem is contained in Δ^{l+n} by Lemma 3.4.
For the second statement of the theorem, let $i \in \{1, ..., m\}$ and let (h, y) be in the intersection of the left hand side of (4) and $g_i \cdot L_i$. Since $(W(h), L(h))$ is special, we have $\dim W(h) < n$. Hence $\dim L_i(h) < n$ by (6). It follows directly that $\dim L_i(z) < n$ for every $z \in \mathbb{R}^l$. □

4. The axiomatization

Let Γ be a multiplicative subgroup of $\mathbb{R}_{>0}$ such that the divisible closure of Γ has the Mann property over $\mathbb{Q}(\tau)$ and Γ is a subset of $\mathbb{Q}(\tau)^{ac}$. Let Δ be the $\mathbb{Q}(\tau)$-linear subspace of $\mathbb{R}_{>0}$ generated by Γ. Further we assume that

$$|\Gamma : \Gamma^{[d]}| < \infty, \text{ for every } d \in \mathbb{N}, \tag{7}$$

where $\Gamma^{[d]}$ is the group of dth powers of Γ. In the rest of this section, axiomatizations of $(\overline{\mathbb{R}}, x^\tau, \tau, \Gamma)$ and $(\overline{\mathbb{R}}, x^\tau, \tau, \Delta)$ will be given.

Note that (7) holds for every multiplicative subgroup of $\mathbb{R}_{>0}$ which has finite rank.

4.1. **Abelian subgroups.** Let G be a multiplicative subgroup of $(M_{>0}, \cdot)$ for some real closed field M. For $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ and $g = (g_1, \ldots, g_n) \in G^n$, we define

$$\chi_k(g) := g_1^{k_1} \cdot ... \cdot g_n^{k_n}.$$

Also, for $m \in \mathbb{Z}$, we will write

$$D_{k,m} := \{g \in G^n : \chi_k(g) \in G^{[m]}\}.$$

Note that $(G^{[m]})^n \subseteq D_{k,m}$. Hence whenever $G^{[m]}$ is of finite index in G we have that $D_{k,m}$ is of finite index in G^n. This implies that both $D_{k,m}$ and $G^n \setminus D_{k,m}$ are finite unions of cosets of $(G^{[m]})^n$. Using the fact that the collection $\{(G^{[m]})^n : m \in \mathbb{N}\}$ is a distributive lattice of subgroups of G^n, we get the following consequence.

Lemma 4.1. *Let $n > 0$, $k_1, \ldots, k_s \in \mathbb{Z}^n$ and $m_1, \ldots, m_t \in \mathbb{N}$. Suppose that $|G : G^{[m_j]}|$ is finite for $j = 1, \ldots, t$. Then every boolean combination of cosets of D_{k_i,m_j} in G^n with $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$ is a finite union of cosets of $(G^{[l]})^n$, where l is the lowest common multiple of $m_1, \ldots, m_t$.*

We say a subgroup H of G is *pure*, if $h \in H^{[n]}$ whenever $h \in G^{[n]}$ for $n \in \mathbb{N}$.
For a pure subgroup H of G and a subset A of G, we define $H_G\langle A\rangle$ as the set of $g \in G$ such that g^n is in the subgroup of G generated by H and A for some $n > 0$; that is there are $h \in H$, $a \in A^t$, and $k \in \mathbb{Z}^t$, such that $g^n = h \cdot \chi_k(a)$. Note that $H_G\langle A\rangle$ is the smallest pure subgroup of G containing A and H.

4.2. **Languages and Mordell-Lang axioms for special pairs.** Let $\mathfrak{L}$ be the language of $(\overline{\mathbb{R}}, x^\tau, \tau)$. We define the language $\mathfrak{L}_\Gamma$ as $\mathfrak{L}$ augmented by a constant symbol $\dot\gamma$ for every $\gamma \in \Gamma$. The $\mathfrak{L}$-structure $(\overline{\mathbb{R}}, x^\tau, \tau)$ naturally becomes a $\mathfrak{L}_\Gamma$-structure by interpreting every $\gamma \in \Gamma$ as $\dot\gamma$. Let T_Γ be the theory of this $\mathfrak{L}_\Gamma$-structure. Finally let $\mathfrak{L}_\Gamma(U)$ be the language $\mathfrak{L}_\Gamma$ expanded by an unary predicate symbol U.

Let W be a variety defined over $\mathbb{Q}(\tau)$ and let L be a basic $\mathbb{Q}(\tau)$-torus. Note that both W and L are $\mathfrak{L}$-$\emptyset$-definable. Further let φ be the $\mathfrak{L}_\Gamma(U)$-formula which defines the set

$$S := \{(g, y) \in \Gamma^l \times \mathbb{R}^n : \ (g, y) \in W \text{ and } (W(g), L(g)) \text{ is special}\}.$$

By Theorem 3.3, there are basic $\mathbb{Q}$-tori $L_1, ..., L_m$ and $\gamma_1, ..., \gamma_m \in \Gamma^{l+n}$ such that S is a subset of the union of $\gamma_1 \cdot L_1, ..., \gamma_m \cdot L_m$ and $\dim L_i(z) < n$ for every $i = 1, ..., m$ and $z \in \mathbb{R}^l$. Let $k_{i,1}, ..., k_{i,s_i} \in \mathbb{Z}^{l+n}$ be such that

$$L_i = \{x \in \mathbb{R}^n \ : \ \chi_{k_{i,j}}(x) = 1, \text{ for } j = 1, ..., s_i\}.$$

The *Mordell-Lang axiom of* (W, L) is defined as the $\mathfrak{L}_\Gamma(U)$-formula $\psi_{(W,L)}$ given by

$$\varphi(x) \to \bigvee_{i=1}^{m} \bigwedge_{j=1}^{s_i} \chi_{k_{i,j}}(x) = \chi_{k_{i,j}}(\gamma_i).$$

4.3. **The theory.** We consider the class of all $\mathfrak{L}_\Gamma(U)$-structure (M, G) satisfying the following axioms:

- (A1) M is a model of T_Γ,
- (A2) G is a dense multiplicative subgroup of M with pure subgroup Γ,
- (A3) $|\Gamma : \Gamma^{[n]}| = |G : G^{[n]}|$, for all $n \in \mathbb{Q}$,
- (A4) $L \cap (G \setminus \{1\})^n = \emptyset$, for every basic $\mathbb{Q}(\tau)$-torus $L \subseteq M^n$ which is not a basic $\mathbb{Q}$-torus.
- (A5) Mordell-Lang axiom $\psi_{W,L}$ for every variety $W \subsetneq M^{l+n}$ over $\mathbb{Q}(\tau)$ and every basic $\mathbb{Q}(\tau)$-torus $L \subseteq M^{l+n}$,
- (A6) the set
$$\bigcap_{i=1}^{m} \{a \in M \ : \ \forall g \in G^l \ f_i(g, b) \neq a\}$$
is dense in M, for all $b \in M^n$ and $\mathfrak{L}$-$\emptyset$-definable functions $f_1, ..., f_m : M^{l+n} \to M$.

One can easily show that there is a first order $\mathfrak{L}_\Gamma(U)$-theory whose models are exactly the structures satisfying (A1)-(A6). Let $T_\Gamma(\Gamma)$ be this $\mathfrak{L}_\Gamma(U)$-theory.

Proposition 4.2. *Assume Condition 2.1. Then* $(\overline{\mathbb{R}}, x^\tau, \tau, \Gamma) \models T_\Gamma(\Gamma)$.

Proof. The axioms (A1)-(A3) hold by definition. Axiom (A5) is implied by Theorem 3.3. Since Γ is a subset of $\mathbb{Q}(\tau)^{ac}$, it is countable and hence (A6) holds for Γ. Finally consider axiom (A4). Let L be a basic $\mathbb{Q}(\tau)$-torus $L \subsetneq \mathbb{R}^n$ which is not a $\mathbb{Q}$-torus. For a contradiction, suppose there is $g \in (\Gamma \setminus \{1\})^n$ such that $g \in L$. Since every element of Γ is algebraic over $\mathbb{Q}(\tau)$ and L is not a $\mathbb{Q}$-torus, we get

$$\mathrm{td}_{\mathbb{Q}(\tau)}(g) + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(g) = 0 + \mathrm{m.dim}_{\mathbb{Q}(\tau)}(g) < \mathrm{m.dim}_{\mathbb{Q}}(g).$$

This contradicts Condition 2.1. □

For an axiomatization of $(\overline{\mathbb{R}}, x^\tau, \tau, \Delta)$, consider the $\mathfrak{L}_\Gamma(U)$-structures (M, G) satisfying

- (A7) G is a dense multiplicative subgroup of M with subgroup Δ,
- (A8) $g^p \in G$, for every $g \in G$ and $p \in \mathbb{Q}(\tau)$.

Let $T_\Gamma(\Delta)$ be the first order $\mathfrak{L}_\Gamma(U)$-theory whose models are exactly the structures satisfying (A1) and (A5)-(A8).

Proposition 4.3. *Assume Condition 2.1. Then* $(\overline{\mathbb{R}}, x^\tau, \tau, \Delta) \models T_\Gamma(\Delta)$.

Among other things, it will be shown in the next section that both $T_\Gamma(\Gamma)$ and $T_\Gamma(\Delta)$ are complete.

5. Quantifier elimination

In this section, the first part of Theorem A and Theorem B is proved. We continue with the notation fixed at beginning of the last section (see page 7). In the following, $\tilde{T}$ is either $T_\Gamma(\Gamma)$ or $T_\Gamma(\Delta)$.

Let $x = (x_1, \ldots, x_m)$ be a tuple of distinct variables. For every $\mathfrak{L}_\Gamma(U)$-formula $\varphi(x)$ of the form

$$\exists y_1 \cdots \exists y_n \bigwedge_{j=1}^{n} U(y_i) \wedge \psi(x, y_1, \ldots, y_n), \tag{8}$$

where $\psi(x, y_1, \ldots, y_n)$ is an $\mathfrak{L}_\Gamma$-formula, let U_φ be a new relation symbol of arity m. Let $\mathfrak{L}_\Gamma(U)^+$ be the language $\mathfrak{L}_\Gamma(U)$ with relation symbols U_φ for every φ of the form (8). Let $\tilde{T}^+$ be the $\mathfrak{L}_\Gamma(U)^+$-theory extending the theory $\tilde{T}$ by axioms

$$\forall x \big(U_\varphi(x) \leftrightarrow \varphi(x)\big),$$

for each φ of the form (8). In order to show the first part of Theorem A and Theorem B, one has to show the following:

Theorem 5.1. *The theory $\tilde{T}^+$ has quantifier elimination.*

The rest of this section will provide a proof of Theorem 5.1. In fact, we will give the proof only for $\tilde{T} = T_\Gamma(\Gamma)$. The case of $T_\Gamma(\Delta)$ can be handled in almost exactly the same way. We will comment on the differences at the end of this section.

5.1. **Main Lemma.** This subsection establishes the main technical lemma used in the proof of Theorem 5.1. Therefor the following instance of Jones and Wilkie [9], Theorem 4.2, is needed.

Proposition 5.2. *Let $M \models T$ and $b \in M, A \subseteq M$. If $b \in \boldsymbol{cl}_T(A)$, then there are $y \in M^n$, a variety W defined over $\mathbb{Q}(\tau)(A)$ and an $\mathbb{Q}(\tau)$-torus L over A such that $(b, y) \in W \cap L$ and*

$$\dim W + \dim L \leq n + 1.$$

Further y can be assumed to be multiplicatively independent over b, A, ie. for every $a \in A^m$

$$m.dim_{\mathbb{Q}}(y/b, a) = n.$$

Lemma 5.3. *Let $(M, G) \models \tilde{T}$ and H be a pure subgroup of G containing all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. Then*

$$\boldsymbol{cl}_T(H) \cap G = H.$$

Proof. The inclusion $H \subset \mathbf{cl}_T(H) \cap G$ is trivial. It is just left to show that whenever $g \in \mathbf{cl}_T(H) \cap G$, then g is also in H. So let $g \in \mathbf{cl}_T(H) \cap G$. By Proposition 5.2, there is $n \in \mathbb{N}$ such that there are $h \in (H \setminus \{1\})^m$, $y \in M^n$, a variety $W \subseteq M^{m+1+n}$ defined over $\mathbb{Q}(\tau)$ and a basic $\mathbb{Q}(\tau)$-torus $L \subseteq M^{m+1+n}$ such that

$$(g, y) \in W(h) \cap L(h),$$

$$\dim W(h) + \dim L(h) \leq n + 1 \tag{9}$$

and

$$\text{m.dim}_{\mathbb{Q}}(y/h', g) = n, \text{ for every } h' \in H^l. \tag{10}$$

Take n minimal with this property.

We will now show that $n = 0$. For a contradiction, suppose that $n > 0$. We first prove that the pair $(W(h, g), L(h, g))$ is special. Towards a contradiction, suppose there are $z \in W(h, g) \cap L(h, g)$, $l < n$ and a projection $\pi : M^n \to M^l$ with

$$\dim W(h, g)(\pi(z)) + \dim L(h, g)(\pi(z)) \geq n - l.$$

Let $W' \subseteq M^{l+1}$ be the variety defined by all polynomial equations over $\mathbb{Q}(\tau)(h)$ which are satisfied by $(g, \pi(z))$ and let $L' \subseteq M^{l+1}$ be the smallest $\mathbb{Q}(\tau)$-torus over h which contains $(g, \pi(z))$. Then

$$\begin{aligned}
&\dim W' + \dim L' \\
&\qquad \leq \dim W(h) + \dim L(h) - \dim W(h, g)(\pi(z)) - \dim L(h, g)(\pi(z)) \\
&\qquad \leq n + 1 - (n - l) = l + 1.
\end{aligned}$$

But this contradicts the minimality of n. Hence $(W(h, g), L(h, g))$ is special. By (A5), there are $\gamma \in \Gamma$ and a basic $\mathbb{Q}$-torus L_0 such that $(h, g, y) \in \gamma \cdot L_0$ and $\dim L_0(h, g) < n$. Hence $\text{m.dim}_{\mathbb{Q}}(y/\gamma, h, g) < n$. This is a contradiction against (10). Hence $n = 0$.

Since $n = 0$, there is a variety $W \subseteq M$ defined over $\mathbb{Q}(\tau)$ and a basic $\mathbb{Q}(\tau)$-torus $L \subseteq M$ such that $g \in W(h) \cap L(h)$ and

$$\dim W(h) + \dim L(h) \leq 1. \tag{11}$$

First consider the case that $\dim W(h) = 1$. By (11), $\dim L(h) = 0$. By (A4) and $(h, g) \in L$, L is a basic $\mathbb{Q}$-torus. Hence $\mathrm{m.dim}_{\mathbb{Q}}(g/h) = 0$. Since H is pure and $g \in G$, we have $g \in H$.
Now consider $\dim W(h) = 0$. By Definition 3.2 of specialness and $(h, g) \in G^{m+1}$, the pair $(W(h, g), L(h, g))$ is special. By (A5), there are a basic $\mathbb{Q}$-torus L_0 and a $\gamma \in \Gamma$ such that $(h, g) \in \gamma \cdot L_0$. As above, we get $g \in H$. $\square$

Corollary 5.4. *Let $(M, G) \models \tilde{T}$ and H be a pure subgroup of G containing all interpretations of the constants $\dot{\gamma}$, where $\gamma \in \Gamma$. If A is $\mathbf{cl}_T$-independent over G, then*

$$\mathbf{cl}_T(A, H) \cap G = H.$$

Proof. H is obviously a subset of $\mathbf{cl}_T(A, H) \cap G$. By Lemma 5.3 it is only left to show that

$$\mathbf{cl}_T(A, H) \cap G \subseteq \mathbf{cl}_T(H) \cap G. \tag{12}$$

So let $g \in \mathbf{cl}_T(A, H) \cap G$ and A' be a minimal subset of A such that $g \in \mathbf{cl}_T(A', H) \cap G$. For a contradiction, suppose that A' is non-empty and let $a \in A'$. By minimality of A', we have $g \notin \mathbf{cl}_T(A' \setminus \{a\}, H)$. But then the Steinitz Exchange Principle implies that $a \in \mathbf{cl}_T(A' \setminus \{a\}, g, H)$. Since $g \in H \subseteq G$, we get that

$$a \in \mathbf{cl}_T(A' \setminus \{a\}, G).$$

But this is a contradiction to the $\mathbf{cl}_T$-independence of A over G. Hence A' is empty and $g \in \mathbf{cl}_T(H) \cap G$. Thus (12) holds. $\square$

5.2. **Back and forth.** Let $(M, G), (M', G')$ be two $(|\Gamma|)^+$-saturated models of $\tilde{T}$. Then M, M' are models of T_Γ. Let $\mathcal{E}$ be the set of all $\mathfrak{L}_\Gamma$-elementary maps from M to M'. Let $\mathcal{S}$ be the set of all $\beta \in \mathcal{E}$ such that there exist

- a finite subset A of M, and a finite subset A' of M',
- a pure subgroup H of G of cardinality at most $|\Gamma|$ and a pure subgroup H' of G' of cardinality at most $|\Gamma|$

such that

(1) β is an $\mathfrak{L}_\Gamma(U)$-isomorphism between $(\mathbf{cl}_T(A, H), H)$ and $(\mathbf{cl}_T(A', H'), H')$,
(2) A is $\mathbf{cl}_T$-independent over G, and A' is $\mathbf{cl}_T$-independent over G' with $\beta(A) = A'$,
(3) Γ is a pure subgroup of H and H'.

By Corollary 5.4, $(\mathbf{cl}_T(A, H), H)$ and $(\mathbf{cl}_T(A', H'), H')$ are $\mathfrak{L}_\Gamma(U)$-substructures of (M, G) and (M', G') respectively. Hence every element of $\mathcal{S}$ is a partial isomorphism between (M, G) and (M', G').

Lemma 5.5. *The set $\mathcal{S}$ is a back-and-forth system.*

Proof. In order to prove this statement, we will show that for every $\beta \in \mathcal{S}$ and every $a \in M$, there is a $\tilde{\beta} \in \mathcal{S}$ such that $\tilde{\beta}$ extends β and $a \in dom(\gamma)$. In fact, this is enough because of the symmetry of the setting.
Let $\beta \in \mathcal{S}$ and $a \in M$. We can assume that $a \notin dom(\beta)$. Further let A, A', H, H' witness that $\beta \in \mathcal{S}$.
<u>Case 1:</u> $a \in G$.

Let $p(x)$ be the $\mathfrak{L}_\Gamma(U)$-type consisting of the $\mathfrak{L}_\Gamma$-type of a over $\mathbf{cl}_T(A,H)$ and for every $h \in H$, $k \in \mathbb{Z}$ and $n > 0$ one of the formulas

$$x^k \cdot h \in G^{[n]}, \tag{13}$$

$$x^k \cdot h \notin G^{[n]}, \tag{14}$$

depending on whether it is true in (M,G) that $a^k h \in G^{[n]}$ or not. Further let p' be the type over $\mathbf{cl}_T(A',H')$ corresponding to p via β. We want to find an $a' \in M'$ such that a' realizes p'. By compactness and saturation of (M',G'), it is enough to show that finitely many formulas of p' can be satisfied. By o-minimality of T, this reduces to find an $a' \in M'$ with

$$(M',G') \models \beta(c) < a' < \beta(d) \wedge \bigwedge_{i=1}^{n} \phi_i(a'), \tag{15}$$

for every $c,d \in \mathbf{cl}_T(A,H)$ with $c < a < d$ and every finite collection of formulas $\phi_1,\dots,\phi_n$ of the form (13) or (14) with $(M,G) \models \bigwedge_{i=1}^n \phi_i(a)$. By Lemma 4.1, the set

$$Y := \{g \in G' \ : \ (M',G') \models \bigwedge_{i=1}^{n} \phi_i(g)\}$$

is a finite union of cosets of $G'^{[s]}$ in G' for some $s \in \mathbb{N}$. Since $G'^{[s]}$ is dense in G', we have that Y is dense in G' as well. Since G' is dense in M', we have that $Y \cap \big(\beta(c),\beta(d)\big)$ is dense in $\big(\beta(c),\beta(d)\big)$. Now take any $a' \in Y \cap \big(\beta(c),\beta(d)\big)$. This a' satisfies (15).

By definition, $H_G\langle a\rangle$ and $H'_{G'}\langle a'\rangle$ are the smallest pure subgroups of G and G' containing $H \cup \{a\}$ and $H' \cup \{a'\}$ respectively. Let $\tilde{\beta}$ be the $\mathfrak{L}_\Gamma$-isomorphism which extends β to $\mathbf{cl}_T(A,H,a)$ and maps a to a'. By conditions (13) and (14) we get for every $h \in G$ that $h \in H_G\langle a\rangle$ if and only if $\tilde{\beta}(h) \in H'_{G'}\langle a'\rangle$. Hence $\tilde{\beta}$ is an isomorphism of $\big(\mathbf{cl}_T(A,H,a), H_G\langle a\rangle\big)$ and $\big(\mathbf{cl}_T(A',H',a'), H'_{G'}\langle a'\rangle\big)$ and $\tilde{\beta} \in \mathcal{S}$.

Case 2: $a \in \mathbf{cl}_T(A,G)$.

Let $g_1,\dots,g_n \in G$ be such that $a \in \mathbf{cl}_T\big(A,\{g_1,\dots,g_n\}\big)$. By applying the previous case n times, we get a $\tilde{\beta} \in \mathcal{S}$ such that $g_1,\dots,g_n \in \operatorname{dom}(\tilde{\beta})$ and $A \subseteq \operatorname{dom}(\tilde{\beta})$. Since $\operatorname{dom}(\tilde{\beta})$ is a model of T_Γ, we have $a \in \operatorname{dom}(\tilde{\beta})$ with $\tilde{\beta} \in \mathcal{S}$.

Case 3: $a \notin \mathbf{cl}_T(A,G)$.

Let C be the cut of a in $\mathbf{cl}_T(A,H)$ and let C' be the corresponding cut of C under β in $\mathbf{cl}_T(A',H')$. By saturation, we can assume that there are $p,q \in M'$ such that every element in the interval (p,q) realizes the cut C'. Let $d \in M^{|A|}$ be the set A written as a tuple. Let $f_1,\dots,f_n$ be $\emptyset$-definable functions in the language $\mathfrak{L}_\Gamma$. By (A6), we know that there exists $b \in (p,q)$ such that for $i = 1,\dots,n$ and every tuple $g_1,\dots,g_l$ of elements of G'

$$f_i(g_1,\dots,g_l,d) \neq b.$$

Thus by saturation, there is an $a' \in (p,q)$ such that $a' \notin \mathbf{cl}_T(A',G')$. Since a' realizes the cut C', there is an $\mathfrak{L}_\Gamma$-isomorphism $\tilde{\beta}$ from $\mathbf{cl}_T(A,a,H)$ to $\mathbf{cl}_T(A',a',H')$ extending β and sending a to a'. Since $a \notin \mathbf{cl}_T(A,G)$ and $a' \notin \mathbf{cl}_T(A',G')$, we get that

$$\mathbf{cl}_T(A,a,H) \cap G = H \text{ and } \mathbf{cl}_T(A',a',H') \cap G' = H'.$$

Since $\beta(H) = H'$ and $\tilde{\beta}$ extends β, we get that $\tilde{\beta}$ is an $\mathfrak{L}_\Gamma(U)$-isomorphism from $(\mathbf{cl}_T(A, a, H), H)$ to $(\mathbf{cl}_T(A', a', H'), H')$ with $\tilde{\beta}(A \cup \{a\}) = A' \cup \{a'\}$. Thus we have that $\tilde{\beta} \in \mathcal{S}$. □

Theorem 5.6. *Assume Condition 2.1. Then $\tilde{T}$ is complete.*

Proof. Let (M, G) and (M', G') be two $|\Gamma|^+$-saturated models of $\tilde{T}$, and let $\mathcal{S}$ be as above. It only remains to show that $\mathcal{S}$ is non-empty. But it is easy to see that the identity map on $\mathbf{cl}_T(\Gamma)$ belongs to $\mathcal{S}$. □

5.3. **Quantifier elimination.** In this subsection Theorem 5.1 is finally proved (see page 9 for the statement).

Proof of Theorem 5.1. Let (M, G) and (M', G') be two $|\Gamma|^+$-saturated models of $\tilde{T}^+$ and let $\mathcal{S}$ be the back-and-forth system between (M, G) and (M', G') constructed above. Also take $a = (a_1, \dots, a_n) \in M^n$ and $b = (b_1, \dots, b_n) \in (M')^n$ satisfying the same quantifier-free $\mathfrak{L}_\Gamma(U)^+$-type. In order to prove quantifier elimination, we just need to find $\tilde{\beta} \in \mathcal{S}$ sending a to b. Without loss of generality, we can assume that $a_1, \dots, a_r$ are maximally independent over G in respect to the pregeometry $\mathbf{cl}_T$. Since a and b have the same $\mathfrak{L}_\Gamma(U)^+$-type, we get that $b_1, \dots, b_r$ are independent over G' in respect to the pregeometry $\mathbf{cl}_T$. Let β be the $\mathfrak{L}_\Gamma$-isomorphism between $\mathbf{cl}_T(\{a_1, \dots, a_r\}, \Gamma)$ and $\mathbf{cl}_T(\{b_1, \dots, b_r\}, \Gamma)$. We will now show that β extends to an isomorphism $\tilde{\beta}$ in the back-and-forth-system $\mathcal{S}$ sending a to b. Let $g_1, \dots, g_l \in G$ be such that $a_{r+1}, \dots, a_n$ are in $\mathbf{cl}_T(\{a_1, \dots, a_r, g_1, \dots, g_l\}, \Gamma)$. Let $p(x_1, \dots, x_l)$ be the $\mathfrak{L}_\Gamma(U)$-type consisting of the $\mathfrak{L}_\Gamma$-type of $(g_1, \dots, g_l)$ over $\mathbf{cl}_T(\{a_1, \dots, a_r\}, \Gamma)$ and for every $k_1, \dots, k_l \in \mathbb{Z}$, $s \in \mathbb{N}$ and $\gamma \in \Gamma$ one of the formulas

$$x_1^{k_1} \cdot \ldots \cdot x_l^{k_l} \cdot \gamma \in G^{[s]}, \tag{16}$$

$$x_1^{k_1} \cdot \ldots \cdot x_l^{k_l} \cdot \gamma \notin G^{[s]}, \tag{17}$$

depending on whether $g_1^{k_1} \cdot \ldots \cdot g_l^{k_l} \cdot \gamma \in G^{[s]}$. Let p' be the type corresponding to p under β. We want to find $h_1, \dots, h_l \in G'$ satisfying p'. By compactness and saturation of (M', G'), it is enough to show that every finite subset of p' can be realized. So let $\psi(x, b_1, \dots, b_r)$ be an $\mathfrak{L}_\Gamma$-formula in p' and $\chi_1(x, b_1, \dots, b_r), \dots, \chi_t(x, b_1, \dots, b_r)$ be finitely many formulas in p' of the form (16) or (17). Put $\chi = \bigwedge_{i=1}^t \chi_i$. By Lemma 4.1, the set

$$Y := \left\{(h_1, \dots, h_l) \in G'^l \ : \ (M', G') \models \chi(h_1, \dots, h_l, b_1, \dots, b_r))\right\}$$

is a finite union of cosets of $(G'^{[s]})^l$ in $(G')^l$ for some $s \in \mathbb{N}$. So the formula $\chi_i(x, b_1, \dots, b_r)$ is equivalent to an atomic $\mathfrak{L}_\Gamma(U)^+$-formula. Hence the formula $\psi \wedge \chi$ is also of this form. Thus

$$\exists y_1 \cdots \exists y_l \bigwedge_{i=1}^{l} U(y_i) \wedge \psi(y_1, \dots, y_l, b_1, \dots, b_r) \wedge \chi(y_1, \dots, y_l, b_1, \dots, b_r)$$

is equivalent to a quantifier-free $\mathfrak{L}_\Gamma(U)^+$-formula. Since $(a_1, \dots, a_r)$ and $(b_1, \dots, b_r)$ have the same quantifier-free $\mathfrak{L}_\Gamma(U)^+$-type, the formula

$$\exists y_1 \cdots \exists y_l \bigwedge_{i=1}^{l} U(y_i) \wedge \psi(y_1, \dots, y_l, b_1, \dots, b_r) \wedge \chi(y_1, \dots, y_l, b_1, \dots, b_r)$$

holds in (M', G'). So p' is finitely satisfiable. Now let $h_1, \dots, h_l \in G'$ realize p'. Then β extends to an $\mathfrak{L}_\Gamma$-isomorphism

$$\tilde{\beta} : \mathbf{cl}_T\big(\{a_1, \dots, a_r, g_1, \dots, g_l\}, \Gamma\big) \to \mathbf{cl}_T\big(\{b_1, \dots, b_r, h_1, \dots, h_l\}, \Gamma\big).$$

By the construction of $g_1, \dots, g_l$ and $h_1, \dots, h_l$, we have that

$$g_1^{k_1} \cdot ... \cdot g_l^{k_l}\gamma \in G^{[s]} \text{ if and only if } h_1^{k_1} \cdot ... \cdot h_l^{k_l}\gamma \in G'^{[s]}$$

for all $k_1, \dots, k_l \in \mathbb{Z}$, $s \in \mathbb{N}$ and $\gamma \in \Gamma$. Hence $\tilde{\beta}$ is an $\mathfrak{L}_\Gamma(U)$-isomorphism of

$$\big(\mathbf{cl}_T(\{a_1, \dots, a_r, g_1, \dots, g_l\}, \Gamma), \Gamma_G\langle g_1, \dots, g_l\rangle\big) \text{ and}$$

$$\big(\mathbf{cl}_T(\{b_1, \dots, b_r, h_1, \dots, h_l\}, \Gamma), \Gamma_{G'}\langle h_1, \dots, h_l\rangle\big).$$

Hence $\tilde{\beta} \in \mathcal{S}$. □

5.4. **Induced structure and open core.** In this subsection it will be shown that every open definable set in $(\overline{\mathbb{R}}, x^\tau, \tau, \Gamma)$ is already definable in the reduct $(\overline{\mathbb{R}}, x^\tau, \tau)$. This establishes the second part of Theorem A. We use the following instance of [7], Theorem 5.2.

Theorem 5.7. *Suppose that for every model $(M, G) \models \tilde{T}$,*

- *for every finite $B \subseteq M$ such that $B \setminus G$ is $\mathbf{cl}_T$-independent over G and*
- *for every set $X \subseteq G^n$ definable in (M, G) with parameters from B,*

the topological closure $\overline{X}$ of X is definable in M over B. Then every open set definable in $(\overline{\mathbb{R}}, x^\tau, \tau, \Gamma)$ is already definable in $(\overline{\mathbb{R}}, x^\tau, \tau)$.

In the remainder it will be shown that the assumption of Theorem 5.7 holds. Therefor let (M, G) be a model of $\tilde{T}$ and let B be a finite subset of M such that $B \setminus G$ is $\mathbf{cl}_T$-independent over G.

Lemma 5.8. *Let $X \subseteq G^n$ be definable in (M, G) with parameters from B. Then X is a finite union of sets of the form*

$$E \cap \bigcup_{i=1}^{l} \gamma_i \cdot (G^{[s]})^n. \tag{18}$$

where $E \subseteq M^n$ is $\mathfrak{L}_\Gamma$-B-definable, $\gamma_1, \dots, \gamma_l \in \Gamma^n$ and $s \in \mathbb{N}$.

Proof. We may assume that (M, G) is a $|\Gamma|^+$-saturated model of $\tilde{T}$. By our assumption, B is a union of a finite set $S \subseteq G$ and a set $A \subseteq M$ which is $\mathbf{cl}_T$-independent over G. Let $\mathcal{S}$ be the back-and-forth system of partial $\mathfrak{L}_\Gamma(U)$-isomorphisms between (M, G) and itself constructed above. Take $g, g' \in G^n$ such that for every $E \subseteq M^n$ $\mathfrak{L}_\Gamma$-definable over B, $\gamma_1, \dots, \gamma_l \in \Gamma^n$ and $s \in \mathbb{N}$ we have that

$$g \in E \cap \bigcup_{i=1}^{l} \gamma_i (G^{[s]})^n \Leftrightarrow g' \in E \cap \bigcup_{i=1}^{l} \gamma_i (sG^{[s]})^n. \tag{19}$$

By Lemma 4.1 and (A3), the collection of finite union of sets of the form (18) is closed under boolean operations. Hence it suffices to show that there is $\beta \in \mathcal{S}$ fixing B such that β maps g to g'. Since g satisfies all $\mathfrak{L}_\Gamma$-formulas over B which are satisfied by g, there is an $\mathfrak{L}_\Gamma$-isomorphism from $\mathbf{cl}_T(g, B, \Gamma)$ to $\mathbf{cl}_T(g', B, \Gamma)$ fixing $B \cup \Gamma$ and mapping g to g'. We now show that $\beta \in \mathcal{S}$. Since $B = S \cup A$, it

is only left to prove that $\beta(\Gamma\langle g, S\rangle) = \Gamma\langle g', S\rangle$. Since β maps g to g' and fixes S, it is enough to show for all $h \in \Gamma_G\langle S\rangle^n$, $k \in \mathbb{Z}^n$ and $s \in \mathbb{N}$ that

$$g \in h \cdot D_{k,s} \text{ if and only if } g' \in h \cdot D_{k,s}.$$

By Lemma 4.1 and (A3), there is are $\gamma_1, \ldots, \gamma_{l_1}, \delta_1, \ldots, \delta_{l_2} \in \Gamma^n$ such that $h \cdot D_{k,s} = \bigcup_{i=1}^{l_1} \gamma_i (G^{[s]})^n$ and $G^n \setminus (h \cdot \gamma D_{k,s}) = \bigcup_{i=1}^{l_2} \delta_i (G^{[s]})^n$. By (19), we have $g \in h \cdot D_{k,s}$ if and only if $g' \in h \cdot D_{k,s}$. Hence $\beta(\Gamma\langle g, S\rangle) = \Gamma\langle g', S\rangle$ and $\beta \in \mathcal{S}$. $\square$

Proposition 5.9. *Let $X \subseteq G^n$ be definable in (M, G) with parameters from B. Then the topological closure $\overline{X}$ of X is definable in M over B.*

Proof. We prove that there is an $\mathfrak{L}_\Gamma$-B-definable set $E \subseteq M^n$ such that X is a dense subset of E. We do this by induction on n. The case $n = 0$ is trivial. So let $n > 0$. By Lemma 5.8 we may assume that there exists an $\mathfrak{L}_\Gamma$-B-definable set E_0 and an $\mathfrak{L}_\Gamma(U)$-$\emptyset$-definable set D_0 which is dense in G^n such that $X = E_0 \cap D_0$. Without loss of generality, we can assume that E_0 is a cell. First consider the case that E_0 is open. Then X is dense in E_0. Now consider the case that there is a projection $\pi : M^n \to M^m$ such that $m < n$ and π is homeomorphism of E_0 onto its image and $\pi(E_0)$ is open. Consider the set

$$X' := \{(g_1, \ldots, g_m) \in G^m \cap \pi(E_0) \ : \ \pi^{-1}(g_1, \ldots, g_m) \in D_0\}.$$

By the induction hypothesis, there is an $\mathfrak{L}_\Gamma$-B-definable subset E_1 of $\pi(E_0)$ such that X' is dense in E_1. By continuity of π^{-1}, the image of X' under π^{-1} is dense in the image of E_1 under π^{-1}. Set $E := \pi^{-1}(E_1)$. Since $X = \pi^{-1}(X')$, we have that X is dense E. $\square$

5.5. **Proof of Theorem B.** As mentioned above, the proof of Theorem B, ie. the case $\tilde{T} = T_\Gamma(\Delta)$, is almost exactly the same as the proof of Theorem A. One only needs to replace 'H is a pure subgroup of G' by 'H is a $\mathbb{Q}(\tau)$-linear subspace of G' in the statement of Lemma 5.3 and the definition of the back-and-forth system $\mathcal{S}$, and adjust the proof of Lemma 5.5 and Theorem 5.1 accordingly.

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